Exercise 76

Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

Solution

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}(36)$$

Use the chain rule to differentiate $y = y(x)$.

$$2x + 8y\frac{dy}{dx} = 0$$

Solve for dy/dx.

$$8y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{4y}$$

At a point of intersection (x_0, y_0), the slope of the tangent line is

$$m = -\frac{x_0}{4y_0}. \tag{1}$$

Use the point-slope formula to get the equation of this line, which passes through the point $(12, 3)$.

$$y - 3 = m(x - 12)$$

$$y - 3 = mx - 12m$$

$$y = mx + (3 - 12m)$$

To determine the points of intersection in terms of m, solve the following system of equations.

$$\begin{cases} y_0 = mx_0 + (3 - 12m) \\ x_0^2 + 4y_0^2 = 36 \end{cases}$$

Substitute the formula for y_0 into the second equation.

$$x_0^2 + 4[mx_0 + (3 - 12m)]^2 = 36$$

$$x_0^2 + 4[m^2x_0^2 + 2m(3 - 12m)x_0 + (3 - 12m)^2] = 36$$

$$(1 + 4m^2)x_0^2 + 8m(3 - 12m)x_0 + [4(3 - 12m)^2 - 36] = 0$$

Use the quadratic formula to solve for x_0.

$$x_0 = \frac{-8m(3-12m) \pm \sqrt{64m^2(3-12m)^2 - 4(1+4m^2)[4(3-12m)^2 - 36]}}{2(1+4m^2)}$$

$$x_0 = \frac{-8m(3-12m) \pm \sqrt{1152m - 1728m^2}}{2(1+4m^2)}$$

$$x_0 = \frac{-8m(3-12m) \pm 24\sqrt{m(2-3m)}}{2(1+4m^2)}$$

$$x_0 = \frac{-4m(3-12m) \pm 12\sqrt{m(2-3m)}}{1+4m^2}$$

The values of y_0 corresponding to these values of x_0 are

$$\begin{cases} x_0 = \dfrac{-4m(3-12m) - 12\sqrt{m(2-3m)}}{1+4m^2} : \quad y_0 = m\left[\dfrac{-4m(3-12m) - 12\sqrt{m(2-3m)}}{1+4m^2}\right] + (3-12m) \\ x_0 = \dfrac{-4m(3-12m) + 12\sqrt{m(2-3m)}}{1+4m^2} : \quad y_0 = m\left[\dfrac{-4m(3-12m) + 12\sqrt{m(2-3m)}}{1+4m^2}\right] + (3-12m) \end{cases},$$

or after simplifying,

$$\begin{cases} x_0 = \dfrac{-4[m(3-12m) + 3\sqrt{m(2-3m)}]}{1+4m^2} : \quad y_0 = \dfrac{3 - 12m - 12m\sqrt{m(2-3m)}}{1+4m^2} \\ x_0 = \dfrac{-4[m(3-12m) - 3\sqrt{m(2-3m)}]}{1+4m^2} : \quad y_0 = \dfrac{3 - 12m + 12m\sqrt{m(2-3m)}}{1+4m^2} \end{cases}.$$

Now substitute each pair of these values for x_0 and y_0 into equation (1).

$$\begin{cases} m = -\dfrac{\frac{-4[m(3-12m)+3\sqrt{m(2-3m)}]}{1+4m^2}}{4\left[\frac{3-12m-12m\sqrt{m(2-3m)}}{1+4m^2}\right]} = \dfrac{m(3-12m) + 3\sqrt{m(2-3m)}}{3 - 12m - 12m\sqrt{m(2-3m)}} \\ m = -\dfrac{\frac{-4[m(3-12m)-3\sqrt{m(2-3m)}]}{1+4m^2}}{4\left[\frac{3-12m+12m\sqrt{m(2-3m)}}{1+4m^2}\right]} = \dfrac{m(3-12m) - 3\sqrt{m(2-3m)}}{3 - 12m + 12m\sqrt{m(2-3m)}} \end{cases}$$

Solve each of these equations for m.

$$\begin{cases} \cancel{m(3-12m)} - 12m^2\sqrt{m(2-3m)} = \cancel{m(3-12m)} + 3\sqrt{m(2-3m)} \\ \cancel{m(3-12m)} + 12m^2\sqrt{m(2-3m)} = \cancel{m(3-12m)} - 3\sqrt{m(2-3m)} \end{cases}$$

Multiplying both sides of one equation by -1 gives the other, so this is actually one equation.

$$-12m^2\sqrt{m(2-3m)} = 3\sqrt{m(2-3m)}$$

$$-12m^2\sqrt{m(2-3m)} - 3\sqrt{m(2-3m)} = 0$$

$$-(12m^2+3)\sqrt{m(2-3m)} = 0$$

$$\sqrt{m(2-3m)} = 0$$

$$m(2-3m) = 0$$

$$m = 0 \quad \text{or} \quad 2-3m = 0$$

$$m = 0 \quad \text{or} \quad m = \frac{2}{3}$$

Both the tangent lines go through $(12, 3)$; therefore, using the point-slope formula, the equations of the tangent lines are

$$y - 3 = 0(x-12) \quad \text{and} \quad y - 3 = \frac{2}{3}(x-12)$$

$$y - 3 = 0 \quad \text{and} \quad y - 3 = \frac{2}{3}x - 8$$

$$y = 3 \quad \text{and} \quad y = \frac{2}{3}x - 5.$$